MATH 320 NOTES, WEEK 7

Recall that we proved that following lemmas:
If $L : V \to W$ is a linear transformation, then

(1) $\ker(L) = \{x \in V \mid L(x) = \vec{0}\}$ is a subspace of V, $nullity(L) = \dim(\ker(L))$.
(2) $\mathrm{ran}(L) = \{L(x) \mid x \in V\}$ is a subspace of W, $rank(L) = \dim(\mathrm{ran}(L))$.
(3) L is one-to-one iff $\ker L = \{\vec{0}\}$.
(4) If $\{L(x_1), ..., L(x_n)\}$ is linearly independent, then $\{x_1, ..., x_n\}$ are linearly independent.
(5) If $span(\{x_1, ..., x_n\}) = V$, then $span(\{L(x_1), ..., L(x_n)\}) = \mathrm{ran}(L)$.

Theorem 1. *Suppose that V is a finite dimensional vector space and $L : V \to W$ is a linear transformation. Then*

$$\dim(V) = nullity(L) + rank(L).$$

Proof. Suppose that $\alpha = \{x_1, ..., x_n\}$ is a basis for $\ker(L)$. Extend α to a basis $\beta = \{x_1, ..., x_k, x_{k+1}, ..., x_n\}$ for V. Let

$$\gamma = \{L(x_{k+1}), ..., L(x_n)\}.$$

We claim that $|\gamma| = n - k$ and that γ is a basis for $\mathrm{ran}(L)$.

Span First we show that $span(\gamma) = \mathrm{ran}(L)$. Since β is a basis for V, we have that $span(\{x_1, ..., x_n\}) = V$, and so, by the last item (5) above, we have that $span(\{L(x_1), ..., L(x_n)\}) = \mathrm{ran}(L)$.

And, since $x_1, ..., x_k \in \ker(L)$, we have that:
$span(\{L(x_1), ..., L(x_k), L(x_{k+1}), ..., L(x_n)\}) = span(\vec{0}, L(x_{k+1}), ..., L(x_n)\}) = span(\gamma)$.

It follows that $span(\gamma) = \mathrm{ran}(L)$.

Linear independence Next we show that γ is linearly independent. Suppose that for some scalars $a_{k+1}, ..., a_n$,

$$a_{k+1}L(x_{k+1}) + ... + a_nL(x_n) = \vec{0}.$$

Then, by linearity,

$$L(a_{k+1}x_{k+1} + ... + a_nx_n) = \vec{0}.$$

So, $a_{k+1}x_{k+1} + ... + a_nx_n \in \ker(L)$. Then since α is a basis for the kernel, for some scalars $b_1, ..., b_k$,

$$a_{k+1}x_{k+1} + ... + a_nx_n = b_1x_1 + ... + b_kx_k,$$

then

$$b_1x_1 + ... + b_kx_k - a_{k+1}x_{k+1} - ... - a_nx_n = \vec{0}.$$

Since β is linearly independent, we have $b_1 = ... = b_k = a_{k+1} = ... = a_n = 0$, which is what we wanted to show.

Note that, since γ is linearly independent, for any $k < i < j \leq n$, $L(x_i) \neq L(x_j)$. (Otherwise $L(x_i) - L(x_j)$ would be a nontrivial linear combination equal to $\vec{0}$.). So, $\dim(\text{ran}(L)) = |\gamma| = n - k$.

In conclusion,

$$nullity(L) + rank(L) = k + (n - k) = n = \dim(V).$$

$\square$

Corollary 2. *Suppose $T : V \to W$ is a linear transformation, V is finite dimensional, and $\dim(V) = \dim(W)$.*

Then, TFAE

1. *T is one-to-one;*
2. *T is onto;*
3. *$rank(T) = \dim(V)$.*

Proof. We have that T is one-to-one iff $\ker T = \{\vec{0}\}$ iff $nullity(T) = 0$ iff $rank(T) = \dim(V) = \dim(W)$ iff $\dim(\text{ran}(T)) = \dim(W)$ iff $\text{ran}(T) = W$ iff T is onto.

$\square$

Theorem 3. *Let V, W be two vector spaces over F, and suppose that $\{v_1, ..., v_n\}$ is a basis for V. Then for any fixed set of vectors $\{w_1, ..., w_n\}$ in W, there is a unique linear transformation $T : V \to W$, such that for all $i \leq i \leq n$,*

$$T(v_i) = w_i.$$

Proof. For the existence, define a linear transformation T as follows. Let $x \in V$. Let $a_1, ..., a_n$ be the unique scalars, such that

$$x = a_1 v_1 + ... + a_n v_n.$$

Set $T(x) = a_1 w_1 + ... + a_n w_n$. Then by definition, for every $1 \leq i \leq n$, $T(v_i) = w_i$.

Next we have to show that T is linear. So, suppose that $x, y \in V$ and $c \in F$. we have to show that $T(cx + y) = cT(x) + T(y)$. Let $a_1, ..., a_n$ and $b_1, ..., b_n$ be the unique scalars, such that

- $x = a_1 v_1 + ... + a_n v_n$,
- $y = b_1 v_1 + ... + b_n v_n$.

Then,

- $T(x) = a_1 w_1 + ... + a_n w_n$,
- $T(y) = b_1 w_1 + ... + b_n w_n$.

So, $cT(x) + T(y) = c(a_1 w_1 + ... + a_n w_n) + b_1 w_1 + ... + b_n w_n =$

$$= (ca_1 + b_1)w_1 + ... + (ca_n + b_n)w_n.$$

Also, we have that

- $cx + y = c(a_1 v_1 + ... + a_n v_n) + (b_1 v_1 + ... + b_n v_n) =$

  $$= (ca_1 + b_1)v_1 + ... + (ca_n + b_n)v_n, \text{ and so,}$$

- $T(cx + y) = (ca_1 + b_1)w_1 + ... + (ca_n + b_n)w_n$.

It follows that $T(cx + y) = cT(x) + T(y)$. So T is a linear transformation.

Next we have to show uniqueness. To that end, suppose that $T, U : V \to W$ are two linear transformations, such that for all $1 \leq i \leq n$,

$$T(v_i) = U(v_i) = w_i.$$

We have to show that $T = U$.

Let $x \in V$ be arbitrary. Let $a_1, ..., a_n$ be the unique scalars, such that $x = a_1v_1 + ... + a_nv_n$. Then, by linearity of T,

$$T(x) = T(a_1v_1 + ... + a_nv_n) = a_1T(v_1) + ... + a_nT(v_n) = a_1w_1 + ... + a_nw_n.$$

And similarly, by linearity of U,

$$U(x) = U(a_1v_1 + ... + a_nv_n) = a_1U(v_1) + ... + a_nU(v_n) = a_1w_1 + ... + a_nw_n.$$

So $T(x) = U(x)$. Since x was arbitrary, it follows that $T = U$. □

Corollary 4. *Let $T, U : V \to W$ be two linear transformations and $\{v_1, ..., v_n\}$ be a basis for V. Suppose that for all $i \leq i \leq n$, $T(v_i) = U(v_i)$. Then $T = U$.*

2.2 The matrix representation of a linear transformation

Recall that given a matrix $A \in M_{k,n}(F)$, multiplication by this matrix, $L_A : F^n \to F^k$, defined by $L_A(x) = Ax$, is a linear transformation.

In this section we will see that every linear transformation $T : V \to W$, where V is finite dimensional, can be viewed as multiplication by a matrix. Since the domain of matrix multiplication is of the form F^n, first we have to find a way to identify V as F^n, where $n = \dim(V)$. For that we need the notion of an ordered basis β for V. Then, for a vector $x \in V$, the notion of a coordinate vector of x relative to β.

Definition 5. *$\beta = \{x_1, ..., x_n\}$ is* **an ordered basis** *for a vector space V, if $\{x_1, ..., x_n\}$ is a basis for V, and the vectors are ordered according to the indices. In other words the order of the vectors matter.*

Example: $\{1, x, x^2\}$ is an ordered basis for $P_2(F)$.

Note that $\{x, 1, x^2\}$ is a *different ordered* basis for $P_2(F)$.

Definition 6. *Let $\beta = \{x_1, ..., x_n\}$ be an ordered basis for V and $x \in V$. The* **coordinate vector of x relative to β** *is*

$$[x_\beta] = \langle a_1, ..., a_n \rangle,$$

where $a_1, ..., a_n$ are the unique scalars in F, such that $x = a_1x_1 + ... + a_nx_n$.

Note that $[x_\beta] \in F^n$.

Examples: Consider $p = 5 + 210x - 3x^2 \in P_2(F)$.

(1) Let $\beta = \{1, x, x^2\}$, then
$$[p]_\beta = \langle 5, 210, -3 \rangle;$$
(2) Let $\alpha = \{x^2, x, 1\}$, then
$$[p]_\alpha = \langle -3, 210, 5 \rangle;$$
(3) Let $\gamma = \{1 + x, x + x^2, x^2\}$. Then
$$[p]_\gamma = \langle 5, 205, -208 \rangle;$$

More examples:

(1) Let $x = \langle a_1, ..., a_n \rangle \in F^n$, and let $e = \{e_1, ..., e_n\}$ be the standard ordered basis. Then
$$[x]_e = x$$
.

(2) Let $x = \langle 4, 0, -2 \rangle \in F^3$ and $\alpha = \{\langle 1, 1, 0 \rangle, \langle 1, -1, 0 \rangle, \langle 0, 0, 2 \rangle\}$. Then
$$[x]_\alpha = \langle 2, 2, -1 \rangle.$$

(3) Let $A = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}$, and let $e = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ be the standard ordered basis. Then
$$[A]_e = \langle 1, 2, -1, 0 \rangle;$$

(4) Let A be as above, and $\beta = \{E_{11}, E_{11+12}, E_{11+21}, E_{22}\}$. Then
$$[A]_\beta = \langle 0, 2, -1, 0 \rangle.$$

The next lemma shows that we can use the coordinate vector representation to identify a vector space V of dimension n with F^n.

Lemma 7. *Suppose V is a vector space over a field F, $\dim(V) = n$, and β is an ordered basis for V. Let $\phi_\beta : V \to F^n$ be given by*
$$\phi_\beta(x) = [x]_\beta.$$
Then ϕ_β is a one-to-one, onto linear transformation.

Proof. Let $\beta = \{x_1, ..., x_n\}$.

The proof that ϕ_β is linear is left is one of the homework exercises.

Now to show that ϕ_β is one-to-one: let $x \in \ker(\phi_\beta)$. Then $\phi_\beta(x) = [x]_\beta = \langle 0, ..., 0 \rangle$. By definition of ϕ_β, that means that $x = 0x_1 + ... + 0x_n = \vec{0}$. Then $\ker(\phi_\beta) = \{\vec{0}\}$, and so ϕ_β is one-to-one. Since $\dim(V) = \dim(F^n) = n$, it also follows that ϕ_β is onto.

□

Definition 8. *Let V, W be vector spaces over F, $\dim(V) = n$, $\dim(W) = k$, $\alpha = \{x_1, ..., x_n\}$ an ordered basis for V, $\beta = \{y_1, ..., y_k\}$ an ordered basis for W, and let $T : V \to W$ be a linear transformation. Define $[T]_\alpha^\beta$ to be the following matrix in $M_{k,n}(F)$: for $1 \leq i \leq n$, the i-th column of $[T]_\alpha^\beta$ is $[T(x_i)]_\beta$.*